Supplementary material for the paper : Breakdown of Tan's relation in lossy one-dimensional Bose gases

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This Supplemental Material contains:

- **App. A:** a derivation of formula (5) in the main text for the short-distance behavior of $g^{(1)}(z)$ in the hard-core limit. We also present a numerical method to calculate the momentum distribution $w(p)$ from the rapidity distribution $\rho(q)$ in the hard-core limit.
- **App. B:** the detailed argument for Eq. (4) in the main text at finite g
- **App. C:** the calculation of the hard-core limit of the product $g^2 g^{(2)}(0)$,
- **App. D:** a derivation of the fact that, under one-body losses and in the hard-core limit, the atom density, momentum density and energy density evolve simply as $n(t) = e^{-Gt}n(0)$, $j(t) = e^{-Gt}j(0)$, $e(t) = e^{-Gt}e(0)$
- **App. E:** detailed calculations in the weakly interacting regime within Bogoliubov theory: the evaluation of the momentum distribution, the effect of losses on the Bogoliubov modes, and the solution of the differential equation (13),
- **App. F:** a brief discussion about the generalization of our results to non-uniform gases.

Appendix A: Momentum distribution in the hard-cord limit

In this section we set $\hbar = m = 1$.

1. Conjecture about $g^{(1)}$ on the lattice

We take a lattice gas of free fermions, with creation/annihilation operators $c_j^\dagger$, c_j ($j \in \mathbb{Z}$) satisfying $\{c_j, c_{j'}^\dagger\} = \delta_{j,j'}$. We consider a translation-invariant Gaussian state characterized by the two-point function $\left\langle c_j^\dagger c_{j'} \right\rangle = \left\langle c_{j-j'}^\dagger c_0 \right\rangle$. We want to study the boson one-body density matrix, which includes a Jordan-Wigner string between the two fermion operators. For $j \geq 0$, it is defined as

$$g^{(1)}_{\rm latt.}(j) := \left\langle c_j^\dagger \prod_{a=1}^{j-1} (-1)^{c_a^\dagger c_a} c_0 \right\rangle, \qquad \text{(A1)}$$

and, for $j < 0$, as $g^{(1)}_{\rm latt.}(j) := g^{(1)}_{\rm latt.}(-j)^*$. We use the following exact formula which gives $g^{(1)}_{\rm latt.}(j)$ as a $j \times j$ Toeplitz determinant [**?**],

$$g^{(1)}_{\rm latt.}(j) = 2^{j-1} \begin{vmatrix} G(1) & G(2) & \dots & G(j) \\ G(0) & G(1) & & \vdots \\ \vdots & & \ddots & G(2) \\ G(2-j) & \dots & G(0) & G(1) \end{vmatrix}, \qquad \text{(A2)}$$

with

$$G(j) = \begin{cases} \left\langle c_j^\dagger c_0 \right\rangle & \text{if } \; j \neq 0 \\ \left\langle c_j^\dagger c_0 \right\rangle - \frac{1}{2} & \text{if } \; j = 0. \end{cases} \qquad \text{(A3)}$$

Let us assume that the fermion two-point function depends on a small parameter $\epsilon > 0$, such that its expansion for $\epsilon \to 0^+$ is of the form (for $j \geq 0$)

$$\left\langle c_j^\dagger c_0 \right\rangle \underset{\epsilon \to 0^+}{=} a_0 \epsilon + a_1 j \epsilon^2 + a_2 j^2 \epsilon^3 + a_3 j^3 \epsilon^4 + O(\epsilon^5), \qquad \text{(A4)}$$

and $\left\langle c_j^\dagger c_0 \right\rangle := \left\langle c_{-j}^\dagger c_0 \right\rangle^*$ if $j < 0$. Here the coefficient a_0 is real, but a_1, a_2, a_3 can be complex. For this fermion two-point function, we want to know the small-ϵ expansion of the boson one-density matrix (A1). Using formula (A2), we have computed that expansion with Mathematica, for small values of j. We find

$$\begin{aligned}
g^{(1)}_{\rm latt.}(1) &\underset{\epsilon \to 0^+}{=} a_0\epsilon + a_1\epsilon^2 + a_2\epsilon^3 + a_3\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(2) &= a_0\epsilon + 2a_1\epsilon^2 + 2^2 a_2\epsilon^3 + (2^3 a_3 - 2(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(3) &= a_0\epsilon + 3a_1\epsilon^2 + 3^2 a_2\epsilon^3 + (3^3 a_3 - 8(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(4) &= a_0\epsilon + 4a_1\epsilon^2 + 4^2 a_2\epsilon^4 + (4^3 a_3 - 20(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(5) &= a_0\epsilon + 5a_1\epsilon^2 + 5^2 a_2\epsilon^4 + (5^3 a_3 - 40(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(6) &= a_0\epsilon + 6a_1\epsilon^2 + 6^2 a_2\epsilon^4 + (6^3 a_3 - 70(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(7) &= a_0\epsilon + 7a_1\epsilon^2 + 7^2 a_2\epsilon^4 + (7^3 a_3 - 112(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5) \\
g^{(1)}_{\rm latt.}(8) &= a_0\epsilon + 8a_1\epsilon^2 + 8^2 a_2\epsilon^4 + (8^3 a_3 - 240(2a_0a_2 - a_1^2))\epsilon^4 + O(\epsilon^5),
\end{aligned}$$

which leads us to the obvious conjecture (for $j \geq 0$):

$$g^{(1)}_{\rm latt.}(j) \underset{\epsilon \to 0^+}{=} a_0\epsilon + a_1 j\epsilon^2 + a_2 j^2\epsilon^3 + [a_3 j^3 - \frac{j(j^2-1)}{3}(2a_0a_2 - a_1^2)]\epsilon^4 + O(\epsilon^5). \qquad \text{(A5)}$$

That calculation is of combinatorial nature, and it is probably possible to prove that formula. A proof for all j is not essential for our purposes though. It is sufficient to know that it holds true for a few different values of j. Below, we use it to infer the short-distance behavior of the one-particle density matrix of the continuous Bose gas in the hard-core limit.

2. Eq. (5) in the main text

We consider a continous gas of hard core bosons in a Gaussian state characterized by its rapidity distribution $\rho(q)$. Namely, if $c^\dagger(x)$, $c(x)$ are the fermion creation/annihilation operators in the continuum, we look at a Gaussian state with a translation-invariant fermion two-point function

$$\left\langle c^\dagger(x)c(x')\right\rangle = \int_{-\infty}^{\infty} e^{-iq(x-x')}\rho(q)dq. \qquad \text{(A6)}$$

Let us look first at the short-distance behavior of $\left\langle c^\dagger(x)c(0)\right\rangle$. When $\rho(q)$ decays sufficiently fast (say, exponentially) at large q, it can be obtained simply by expanding the exponential in the integral,

$$\left\langle c^\dagger(x)c(0)\right\rangle \underset{x\to 0}{=} q_0 - iq_1 x - q_2 x^2 + iq_3 x^3 + O(x^4). \qquad \text{(A7)}$$

with $q_a = \int \frac{q^a}{a!}\rho(q)dq$. When $\rho(q)$ decays as a power-law, this expansion breaks down, which is reflected in the fact that the coefficients q_a are infinite for a large enough. From now on we assume that $\rho(q) \simeq \frac{C_{\rm r}}{q^4}$ for $q \to \pm\infty$. The correct small-x expansion is then

$$\left\langle c^\dagger(x)c(0)\right\rangle \underset{x\to 0}{=} q_0 - iq_1 x - q_2 x^2 + iq_3 x^3 + \frac{\pi C_{\rm r}}{6}|x|^3 + O(x^4). \qquad \text{(A8)}$$

Here the coefficient q_3 is finite because the two divergences in the integral $\int q^3/q^4 dq$ when $q \to \pm\infty$ cancel. To obtain the term $\frac{\pi C_{\rm r}}{6}|x|^3$, one can for instance write $\rho(q)$ as $(\rho(q) - \frac{C_{\rm r}}{4+q^4}) + \frac{C_{\rm r}}{4+q^4}$. The first term does not have a tail, so it has an expansion of the form (A7), while the Fourier transform of the second term is evaluated straightforwardly and is $\frac{\pi C_{\rm r}}{4}e^{-|x|}(\cos|x| + \sin|x|) \simeq \frac{\pi C_{\rm r}}{4}(1 - x^2 + \frac{2}{3}|x|^3 + \dots)$.

Now let us turn to the boson one-particle density matrix $g^{(1)}(x)$. We regard $g^{(1)}(x)$ as the continuum limit of $g^{(1)}_{\rm latt.}(j)$ when the lattice spacing ϵ is much smaller than the inverse density of particles $1/q_0$. Namely, for $x \in \epsilon\mathbb{Z}$,

$$g^{(1)}(x) \underset{\epsilon q_0 \ll 1}{\simeq} \frac{1}{q_0\epsilon} g^{(1)}_{\rm latt.}(x/\epsilon). \qquad \text{(A9)}$$

This identification must hold provided that the lattice fermion two-point function corresponds to a discretization of the continuous one. For instance we can take

$$\left\langle c_j^\dagger c_0\right\rangle := \epsilon\left\langle c^\dagger(j\epsilon)c(0)\right\rangle \qquad \text{(A10)}$$
$$= q_0\epsilon - iq_1 j\epsilon^2 - q_2 j^2\epsilon^3 + iq_3 j^3\epsilon^3 + \frac{\pi C_{\rm r}}{6}|j|^3\epsilon^3 + O(a^4).$$

We are interested in the behavior of $g^{(1)}(x)$ for small $x > 0$. We have two small parameters: x and the lattice spacing ϵ (or, equivalently, the dimensionless xq_0 and ϵq_0). Let us consider a smooth function $F(\epsilon, x)$, $\epsilon > 0, x > 0$, which coincides with $\frac{1}{q_0\epsilon} g^{(1)}_{\rm latt.}(x/\epsilon)$ for $x \in \epsilon\mathbb{N}$. Notice that $F(0,x) = g^{(1)}(x)$. $F(\epsilon, x)$ should have a double-expansion in the two small parameters,

$$F(\epsilon, x) = \sum_{l\geq 0, m\geq 0} \alpha_{l,m}\epsilon^l x^m. \qquad \text{(A11)}$$

We can use Eq. (A5), with $a_0 = q_0$, $a_1 = -iq_1$, $a_2 = -q_2$, $a_3 = iq_3 + \frac{\pi C_{\rm r}}{6}$, to fix the first few coefficients $\alpha_{l,m}$. Indeed, for fixed j,

$$\frac{1}{q_0\epsilon} g^{(1)}_{\rm latt.}(j) = F(\epsilon, j\epsilon) = \sum_{l\geq 0, m\geq 0} \alpha_{l,m} j^m \epsilon^{l+m}, \qquad \text{(A12)}$$

so when one expands both sides for small ϵ, the identification of the terms of order $O(\epsilon^{l+m})$ gives

$$\begin{aligned} 1 &= \alpha_{0,0} \\ -i\frac{q_1}{q_0}j &= \alpha_{1,0} + \alpha_{0,1}j \\ -\frac{q_2}{q_0}j^2 &= \alpha_{2,0} + \alpha_{1,1}j + \alpha_{0,2}j^2 \\ (i\frac{q_3}{q_0} + \frac{\pi C_{\rm r}}{6q_0})j^3 + \frac{j(j^2-1)}{3}\frac{2q_0q_2 - q_1^2}{q_0} &= \alpha_{3,0} + \alpha_{2,1}j + \alpha_{1,2}j^2 + \alpha_{0,3}j^3. \end{aligned}$$

Since this holds for several values of j, we get linearly independent equations that fix all the coefficients. In particular, we find $\alpha_{0,1} = -i\frac{q_1}{q_0}$, $\alpha_{0,2} = -\frac{q_2}{q_0}$, $\alpha_{0,3} = i\frac{q_3}{q_0} + \frac{\pi}{6q_0}[C_{\rm r} + \frac{4}{\pi}(q_0q_2 - q_1^2/2)]$.

The continuous one-particle density matrix $g^{(1)}(x)$ is given by $F(0,x)$, so we obtain

$$g^{(1)}(x) \underset{x\to 0^+}{=} 1 - i\frac{q_1}{q_0}x - \frac{q_2}{q_0}x^2 + i\frac{q_3}{q_0}x^3 + \frac{\pi}{6q_0}[C_{\rm r} + \frac{4}{\pi}(q_0q_2 - q_1^2/2)]x^3 + O(x^4). \qquad \text{(A13)}$$

Since $g^{(1)}(-x) = g^{(1)}(x)^*$, we see that we also have

$$g^{(1)}(x) \underset{x\to 0^-}{=} 1 - i\frac{q_1}{q_0}x - \frac{q_2}{q_0}x^2 + i\frac{q_3}{q_0}x^3 - \frac{\pi}{6q_0}[C_{\rm r} + \frac{4}{\pi}(q_0q_2 - q_1^2/2)]x^3 + O(x^4). \qquad \text{(A14)}$$

Thus, our final result for the short-distance behavior of the one-particle density matrix is

$$g^{(1)}(x) \underset{x\to 0}{=} 1 - i\frac{q_1}{q_0}x - \frac{q_2}{q_0}x^2 + i\frac{q_3}{q_0}x^3 + \frac{\pi}{6q_0}[C_{\rm r} + \frac{4}{\pi}(q_0q_2 - q_1^2/2)]|x|^3 + O(x^4). \qquad \text{(A15)}$$

This is our formula (5) in the main text. The coefficient $\frac{4}{\pi}(q_0q_2 - q_1^2/2)$ is the contact density $C_{\rm c}$ in the hard-core limit. This is easily shown by combining formula (3) in the main text with $\lim_{g\to\infty} q_0^2 g^2 g^{(2)}(0) = 8[q_0q_2 - q_1^2/2]$ (in units with $m = \hbar = 1$), see the Appendix C below.

3. Numerical evaluation of the momentum distribution $w(p)$ from the rapidity distribution $\rho(q)$

We have also studied the momentum distribution numerically in the hard-core limit, by evaluating the momentum distribution $w(p)$ of hard-core bosons as a functional of their rapidity distribution $\rho(q)$. Here we explain how we implement that procedure. In this section we set $\hbar = m = 1$. We exploit formulas (14)-(15) of Ref. [?], which gives the one-body density matrix as follows:

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle = \sum_{i,j=0}^{\infty} \varphi_i(x)\sqrt{n_i}Q_{ij}(x,y)\sqrt{n_j}\varphi_j^*(y), \tag{A16}$$

where the $\varphi_i(x)$ ($i = 0,\dots,\infty$) are the single-particle eigenfunctions of the Schrödinger operator for an infinite system in an external potential, $-\hbar^2/(2m)\partial_x^2 + V(x)$, and $n_i \in [0,1]$ is the occupation of each orbital. In Ref. [?], it is assumed that the n_i are the occupations of a Gibbs ensemble at a given temperature and chemical potential. But Eq. (A16) is more general, and it holds true for any occupations, corresponding to a Generalized Gibbs Ensemble. The semi-infinite matrix $Q(x,y)$ is defined as $Q(x,y) = (P^{-1})^T \det P$, with

$$P_{ij}(x,y) = \delta_{ij} - 2\,\mathrm{sign}(y-x)\sqrt{n_i n_j}\int_x^y \phi_i(z)\phi_j^*(z)dz. \tag{A17}$$

We stress that this formula is based on the mapping from hard-core bosons to free fermions, and that it works for an infinite system. In principle, it does not apply to a finite system with periodic boundary conditions. The reason is that hard-core bosons with periodic boundary conditions map to periodic/anti-periodic boundary conditions for the fermions, depending on the whether the total number of fermions is odd/even respectively. Since formula (A16) works for arbitrary occupation numbers, the parity of the number of fermions is not fixed (unless all n_i are equal to 0 or 1).

However, the one-body density matrix typically decays quickly with the distance $|x-y|$. Moreover, we are mostly interested in its short-distance behavior, because this is what fixes the large-p tail of the momentum distribution. Therefore, we can work with $x, y \in [-L/2, L/2]$ with *periodic boundary conditions for the fermions* as long as L is large enough. Thus, we can use plane waves $\varphi_j(x) = e^{iq_j x}/\sqrt{L}$ with $q_j \in 2\pi\mathbb{Z}/L$, such that

$$\left\langle \Psi^\dagger(x)\Psi(0)\right\rangle \underset{L\to\infty}{=} \frac{2\pi}{L}\sum_{q_i,k_j\in\frac{2\pi}{L}\mathbb{Z}} e^{iq_i x}\sqrt{\rho(q_i)\rho(k_j)}Q_{ij}(x,0). \tag{A18}$$

Here we have used the fact that the occupation of each fermionic mode is given by the rapidity density, $n_i = 2\pi\rho(q_i)$. In practice, we numerically evaluate the right hand side of Eq. (A18) by truncating the sum, using a finite set of orbitals $q_i \in \{-\frac{2\pi}{L}M,\dots,-\frac{2\pi}{L},0,\frac{2\pi}{L},\dots,\frac{2\pi}{L}M\}$ for large enough M.

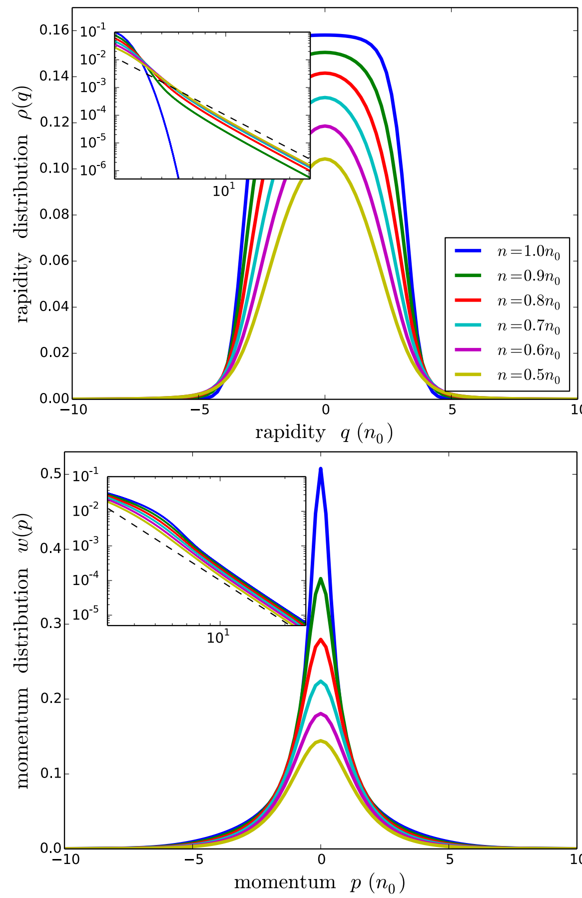

FIG. 1. Top: rapidity distribution in the hard-core limit, given by Eq. (9) in the main text. The initial rapidity distribution $\rho_0(q)$ (blue curve) is the thermal distribution at temperature $T = 1.02n_0^2$ and chemical potential $\mu = 5T$. The other curves are the rapidity distributions after some fraction (10%, 20%, ..., 50%) of the atoms have been lost. The inset shows a zoom on the tails of $\rho(q)$ in logarithmic scale; the black dashed line is the $1/q^4$ curve. In the initial state, $\rho_0(q)$ decays as a Gaussian, but at later times $\rho(q)$ has a $\sim 1/q^4$ tail. Bottom: the corresponding momentum distributions, obtained from our numerical procedure. The inset shows a zoom on the tails of $w(p)$ in logarithmic scale; the black dashed line is the $1/p^4$ curve.

Finally, the momentum distribution is obtained by numerically evaluating the Fourier transform

$$w(p) = \frac{1}{2\pi}\int e^{ipx}\left\langle \Psi^\dagger(x)\Psi(0)\right\rangle dx. \tag{A19}$$

With this method, we obtain the momentum distribution $w(p)$ accurately for $1/L \ll |p| < 2\pi M/L$. In Fig. 1 we show the momentum distribution obtained for rapidity distributions corresponding to Eq. (9) in the main text, for an initial thermal distribution at temperature $T = 1.02n_0^2$ and chemical potential $\mu = 5T$, after some fraction of the atoms have been lost (n_0 is the initial density of atoms). These results are obtained with $L = 31/n_0$ and $M = 125$, so they are accurate for

$0.03n_0 \ll |p| < 25n_0$. This is enough to observe the $1/p^4$ tail (see the inset of Fig. 1, bottom).

In practice, to extract the amplitude of tail C, we use the values of $f(p) := p^4 w(p)$ inside a window $p \in [p_{\min}, p_{\max}]$ where $p_{\min}$ is large enough such that one focuses on the tail, and $p_{\max}$ is small enough so that we avoid the effects of the truncation of the basis of orbitals. We then fit these values with a function $C/p^4 + \alpha_1/p^5 + \alpha_2/p^6$ to extract the coefficient C. This gives us access to C, within an error bar that is typically around $\sim 4\%$.

Alternatively, the amplitude C can be extracted directly from the short-distance behavior of $\langle \Psi^\dagger(x)\Psi(0)\rangle$. Numerically, this is more efficient because one does not have to compute the two-point function for many values of x to evaluate the Fourier transform. One needs only a few values in a small interval $[0, \varepsilon]$, where ε is chosen as some fraction of the inverse density $1/n_0$ (we choose $\varepsilon = 0.25/n_0$). Then we fit these values with a polynomial of the form $\langle \Psi^\dagger(x)\Psi(0)\rangle = n_0 + \alpha_2 x^2 + \frac{\pi C}{6} x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6$, which gives us access to C. The precision of this procedure is higher, and we obtain C with an error of order 0.5%. This is mainly due to the fact that, since we need to compute less points, we can use much larger numbers of orbitals in our truncated sum (A18). We use ~ 6000 orbitals (corresponding to $M \sim 3000$, compared to $M = 125$ above).

We find that the amplitude C obtained with this method always satisfies Eq. (4) in the main text.

Appendix B: Detailed argument for Eq. (4) in the main text at finite g

Here we elaborate on the derivation of the formula $C = C_{\rm c} + C_{\rm r}$ sketched in the main text. The main physical intuition behind this argument is that Bethe quasiparticles with large rapidities λ must correspond to atoms with large momenta $p \simeq \lambda$. We start by making that intuition more precise at the level of Bethe states. In this section we set $m = \hbar = 1$.

1. Preliminary: factorization of Bethe states

Let $\boldsymbol{\lambda}_N = \{\lambda_1, \dots, \lambda_N\}$ be a set of rapidities, with

$$\lambda_1 < \cdots < \lambda_N, \tag{B1}$$

that satisfies the Bethe equations (see below and Ref. [?]). Let $|\boldsymbol{\lambda}_N\rangle$ be the corresponding Bethe state, whose wavefunction is [?]

$$\begin{aligned} &\langle 0| \Psi(x_1) \dots \Psi(x_N) |\boldsymbol{\lambda}\rangle \\ &\propto \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{1 \le a < b \le N} \left(\lambda_{\sigma(b)} - \lambda_{\sigma(a)} - ig\,\mathrm{sgn}(x_b - x_a)\right) \\ &\qquad \times e^{i \sum_a x_a \lambda_{\sigma(a)}}. \end{aligned} \tag{B2}$$

Now let us assume that the largest rapidity is separated from the other ones by an interval much larger than g,

$$|\lambda_N - \lambda_{N-1}| \gg g. \tag{B3}$$

Then we argue that

$$|\boldsymbol{\lambda}_N\rangle \simeq \Psi^\dagger_{\lambda_N} |\boldsymbol{\lambda}_{N-1}\rangle, \tag{B4}$$

where $\Psi^\dagger_p = \frac{1}{\sqrt{L}} \int_0^L e^{ipx} \Psi^\dagger(x) dx$ is the Fourier mode of the boson creation operator $\Psi^\dagger(x)$. This is physically clear: if one boson has very large momentum $p \simeq \lambda_N$, then its interaction with the other $N-1$ bosons is almost suppressed. So the eigenstate must be a tensor product '$\Psi^\dagger_{\lambda_N} |0\rangle \otimes |\boldsymbol{\lambda}_{N-1}\rangle$'. More formally, this is seen directly at the level of Eq. (B2): assuming (B3), we have

$$\begin{aligned} &\langle 0| \Psi(x_1) \dots \Psi(x_N) |\boldsymbol{\lambda}\rangle \\ &\propto \sum_{\sigma \in S_N} (-1)^{|\sigma|} (-1)^{N - \sigma^{-1}(N)} \\ &\prod_{a<b, \sigma(a) \neq N, \sigma(b) \neq N} \left(\lambda_{\sigma(b)} - \lambda_{\sigma(a)} - ig\,\mathrm{sgn}(x_b - x_a)\right) e^{i \sum_a x_a \lambda_{\sigma(a)}}. \end{aligned}$$

We set $d = \sigma^{-1}(N)$ and $\sigma' = \sigma \circ \tau_{dN}$ where τ_{ij} is the transposition $i \leftrightarrow j$, such that $\sigma'(N) = N$. Then we can sum over $d \in \{1, \dots, N\}$ and $\sigma' \in S_{N-1}$ separately. After some straightforward manipulations of the indices, this gives

$$\begin{aligned} &\langle 0| \Psi(x_1) \dots \Psi(x_N) |\boldsymbol{\lambda}\rangle \\ &\propto \sum_{d=1}^N e^{i x_d \lambda_N} \sum_{\sigma' \in S_{N-1}} (-1)^{|\sigma'|} \\ &\prod_{1 \le a < b \le N-1} \left(\lambda_{\sigma(b)} - \lambda_{\sigma(a)} - ig\,\mathrm{sgn}(x_{\tau_{dN}(b)} - x_{\tau_{dN}(a)})\right) \\ &\qquad \times e^{i \sum_{a=1}^{N-1} x_{\tau_{dN}(a)} \lambda_{\sigma(a)}}, \end{aligned}$$

so that we recognize

$$\begin{aligned} &\langle 0| \Psi(x_1) \dots \Psi(x_N) |\boldsymbol{\lambda}\rangle \\ &\quad = \mathcal{S} \cdot e^{i x_N \lambda_N} \langle 0| \prod_{1 \le j \le N-1} \Psi(x_j) |\boldsymbol{\lambda}_{N-1}\rangle, \end{aligned} \tag{B5}$$

where $\mathcal{S}$ is the symmetrizer over all indices of an N-variable function, i.e. $\mathcal{S} \cdot f(x_1, \dots, x_N) := \frac{1}{N!} \sum_{\sigma \in S_N} f(x_{\sigma(1)}, \dots, x_{\sigma(N)})$. Eq. (B5) is nothing but the first-quantized form of Eq. (B4).

Moreover, under the assumption (B3), λ_N becomes independent from the other rapidities at the level of the Bethe equations. Namely, the N equations [?]

$$e^{i\lambda_a L} = \prod_{1 \le b \le N, b \neq a} \frac{\lambda_a - \lambda_b + ig}{\lambda_a - \lambda_b - ig}, \qquad a = 1, \dots, N \tag{B6}$$

become, assuming (B3),

$$\begin{aligned} &e^{i\lambda_a L} = \prod_{1 \le b \le N-1, b \neq a} \frac{\lambda_a - \lambda_b + ig}{\lambda_a - \lambda_b - ig}, \qquad a = 1, \dots, N-1, \\ &e^{i\lambda_N L} = 1. \end{aligned} \tag{B7}$$

Clearly, if one has more rapidities that are widely separated,

$$|\lambda_N-\lambda_{N-1}|, |\lambda_{N-1}-\lambda_{N-2}|, \dots, |\lambda_{N-M+1}-\lambda_{N-M}| \gg g, \tag{B8}$$

then one gets

$$|\boldsymbol{\lambda}_N\rangle \simeq \Psi^\dagger_{\lambda_N}\Psi^\dagger_{\lambda_{N-1}}\dots\Psi^\dagger_{\lambda_{N-M+1}}|\boldsymbol{\lambda}_{N-M}\rangle, \tag{B9}$$

in the same sense as above. This simply follows by induction on M.

2. Model of independent cells

We consider the following model. We take a gas in a very large box of size L. We assume that it has a finite correlation length ξ, so that we can divide it into m small independent cells containing $N^{(j)}$ particles (with a total particle number $N=\sum_{j=1}^m N^{(j)}$), and of length $\ell^{(j)}$ (of order a few times the correlation length ξ). We further assume that the state within each cell may be represented by a single eigenstate for a small periodic system of size $\ell^{(j)}$. The eigenstate in the $j^{\rm th}$ cell is a Bethe state with rapidities $\lambda^{(j)}_1<\dots<\lambda^{(j)}_{N^{(j)}}$, and the rapidity distribution in the full system is taken as the sum of the rapidities in all the cells,

$$\rho(\lambda) := \frac{1}{L}\sum_{j=1}^m\left(\sum_{a=1}^{N^{(j)}}\delta(\lambda-\lambda^{(j)}_a)\right). \tag{B10}$$

In the $m\to\infty$ limit (which implies $L\to\infty$ since we are working with cells of fixed size of order ξ), Eq. (B10) becomes a smooth rapidity distribution. We assume that $\rho(\lambda)$ decays as $C_{\rm r}/\lambda^4$ for large λ.

Now, within the framework of this model, we derive Eq. (4) of the main text. We start by selecting a cutoff Λ large enough so that the following conditions are satisfied:

1. Λ is much larger than the typical width of the distribution $\rho(\lambda)$, so that for $\lambda>\Lambda$, one is really in the tail of the distribution: $\rho(\lambda)\simeq C_{\rm r}/\lambda^4$ for any $\lambda>\Lambda$,
2. $\Lambda\gg g$
3. $\Lambda^4\gg\xi C_{\rm r} g$.

For a cell j, let $M^{(j)}$ be the number of rapidities larger than Λ ($M^{(j)}$ can be zero). Since the rapidities are ordered we have $\lambda^{(j)}_{N^{(j)}-M^{(j)}}<\Lambda<\lambda^{(j)}_{N^{(j)}-M^{(j)}+1}$ when $M^{(j)}>0$. Similarly, we can define $\bar{M}^{(j)}$, the number of rapidities smaller than $-\Lambda$. Because of condition 1., $M^{(j)}$ and $\bar{M}^{(j)}$ can be estimated to be of order

$$M^{(j)} = \ell^{(j)}\int_\Lambda^\infty \rho_{>\Lambda}(\lambda)d\lambda \sim \frac{\ell^{(j)}C_{\rm r}}{\Lambda^3}\sim\frac{\xi C_{\rm r}}{\Lambda^3}. \tag{B11}$$

There are two cases: either this is much smaller than one, or it is larger than one, depending on whether it is condition 2. or 3. that prevails.

If $\xi C_{\rm r}<g^3$, then condition 2. is more restrictive. Condition 2. implies that $\frac{\xi C_{\rm r}}{\Lambda^3}\ll 1$. In that case, we can assume that, in each cell j, $M^{(j)}$ is either zero or one. In the case when $M^{(j)}$ is one, the largest rapidity $\lambda^{(j)}_{N^{(j)}}$ is distributed with a probability $p(\lambda)\simeq\frac{1}{\lambda^4}/\int_\Lambda^\infty\frac{du}{u^4}$, so its distance to all the other rapidities is typically of order Λ. Consequently, condition 2. implies

$$|\lambda^{(j)}_{N^{(j)}}-\lambda^{(j)}_{N^{(j)}-1}|\gg g. \tag{B12}$$

If $\xi C_{\rm r}>g^3$, then condition 3. is more restrictive. Condition 3. does not put a constraint on $M^{(j)}$. [This is because it leads to $\frac{\xi C_{\rm r}}{\Lambda^3}\ll\Lambda/g$, which is automatically satisfied because Λ/g is very large.] In that case there can be several rapidities larger than Λ in each cell j. In an interval $[\lambda,\lambda+\Delta\lambda]$ (with $\lambda>\Lambda$), there are typically $\xi\rho_{>\Lambda}(\lambda)\Delta\lambda\simeq\frac{\xi C_{\rm r}}{\lambda^4}\Delta\lambda$ rapidities, so the typical spacing between two rapidities is $\sim\lambda^4/(\xi C_{\rm r})>\Lambda^4/(\xi C_{\rm r})$. Then condition 3. implies

$$|\lambda^{(j)}_{N^{(j)}}-\lambda^{(j)}_{N^{(j)}-1}|,\dots,|\lambda^{(j)}_{N^{(j)}-M^{(j)}+1}-\lambda^{(j)}_{N^{(j)}-M^{(j)}}|\gg g. \tag{B13}$$

So, in both cases, we find that the $M^{(j)}$ rapidities larger than Λ are separated from the other rapidities by an interval that is large compared to g. Whenever $M^{(j)}>1$, those $M^{(j)}$ rapidities are also well separated from one other. The same discussion applies to the $\bar{M}^{(j)}$ rapidities smaller than $-\Lambda$.

We can then apply the analysis of the previous subsection in each cell j. The Bethe state $\left|\boldsymbol{\lambda}^{(j)}_{N^{(j)}}\right\rangle$ factorizes:

$$\begin{aligned}\left|\boldsymbol{\lambda}^{(j)}_{N^{(j)}}\right\rangle &\simeq \Psi^\dagger_{\lambda^{(j)}_{N^{(j)}}}\dots\Psi^\dagger_{\lambda^{(j)}_{N^{(j)}-M^{(j)}+1}}\\ &\quad\times\Psi^\dagger_{\lambda^{(j)}_1}\dots\Psi^\dagger_{\lambda^{(j)}_{\bar{M}^{(j)}}}\left|\boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1,N^{(j)}-M^{(j)}}\right\rangle,\end{aligned} \tag{B14}$$

where $\left|\boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1,N^{(j)}-M^{(j)}}\right\rangle$ is the Bethe state with rapidities $\{\lambda^{(j)}_{\bar{M}^{(j)}+1},\lambda^{(j)}_{\bar{M}^{(j)}+2}\dots,\lambda^{(j)}_{N^{(j)}-M^{(j)}}\}$. The momentum distribution in the cell j is then given by

$$\begin{aligned}\langle\boldsymbol{\lambda}_{N^{(j)}}|\,\Psi^\dagger_p\Psi_p\,|\boldsymbol{\lambda}_{N^{(j)}}\rangle &\simeq\\ &\left\langle\boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1,N^{(j)}-M^{(j)}}\right|\Psi^\dagger_p\Psi_p\left|\boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1,N^{(j)}-M^{(j)}}\right\rangle\\ &+\sum_{a=1}^{\bar{M}^{(j)}}\delta(p-\lambda^{(j)}_a)+\sum_{a=M^{(j)}+1}^{N^{(j)}}\delta(p-\lambda^{(j)}_a),\end{aligned} \tag{B15}$$

where $\Psi^\dagger_p$ creates a boson in the cell j with momentum p. Summing over the cells and taking the $m\to\infty$ limit, we find the total momentum distribution

$$
\begin{aligned}
w(p) &:= \frac{1}{L}\sum_{j=1}^{m} \langle \boldsymbol{\lambda}_{N^{(j)}} |\, \Psi_p^\dagger \Psi_p \,| \boldsymbol{\lambda}_{N^{(j)}} \rangle \\
&\simeq \frac{1}{L}\sum_{j=1}^{m} \left\langle \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right| \Psi_p^\dagger \Psi_p \left| \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right\rangle \\
&\quad + \frac{1}{L}\sum_{j=1}^{m} \left(\sum_{a=1}^{\bar{M}^{(j)}} \delta(p-\lambda_a^{(j)}) + \sum_{a=M^{(j)}+1}^{N^{(j)}} \delta(p-\lambda_a^{(j)}) \right).
\end{aligned} \tag{B16}
$$

In this second term, we recognize the tail of the rapidity distribution (B10). More precisely, we can split the distribution (B10) into two terms $\rho_{<\Lambda}(\lambda) := \rho(\lambda)\theta(|\Lambda| - \lambda)$ and $\rho_{>\Lambda}(\lambda) := \rho(\lambda)\theta(\lambda - |\Lambda|)$, where $\theta(u) = 1$ if $u \geq 0$ and $\theta(u) = 0$ otherwise. Then the second term in Eq. (B16) is equal to $\rho_{>\Lambda}(\lambda) \simeq \frac{C_{\rm r}}{\lambda^4}\theta(|\lambda| - \Lambda)$. The first term in (B16) is the momentum distribution $w_{<\Lambda}(p)$ evaluated in the macrostate with rapiditity distribution $\rho_{<\Lambda}(\lambda)$.

Thus we arrive at

$$
\begin{aligned}
w(p) &\simeq w_{<\Lambda}(p) + \rho_{>\Lambda}(p) \\
&\underset{|p|\to\infty}{\simeq} \frac{C_{{\rm c},<\Lambda}}{p^4} + \frac{C_{\rm r}}{p^4}.
\end{aligned} \tag{B17}
$$

The term $C_{{\rm c},<\Lambda}/p^4$ comes from Tan's relation, which is valid because the rapidity distribution $\rho_{<\Lambda}(\lambda)$ does not have tails. Notice that this gives the contact density $C_{{\rm c},<\Lambda}$ evaluated in that state, as opposed to the contact density $C_{\rm c}$ evaluated in the macrostate with the initial rapidity distribution $\rho(\lambda)$.

Finally, we show that the contact density $C_{{\rm c},<\Lambda}$ is actually equal to $C_{\rm c}$. To obtain the contact density, we apply the Hellmann-Feynman theorem independently to each cell. We rely again on the factorization of the Bethe state (B14), and on the fact that the Bethe equations for the $M^{(j)} + \bar{M}^{(j)}$ rapidities outside $[-\Lambda, \Lambda]$ decouple, as in Eq. (B7). The fact that the Bethe equations decouple for those rapidities implies that they no longer vary with g, so their derivative w.r.t g vanishes. Thus we have

$$
\begin{aligned}
&\frac{\partial}{\partial g} \langle \boldsymbol{\lambda}_{N^{(j)}} |\, H \,| \boldsymbol{\lambda}_{N^{(j)}} \rangle \\
&\simeq \frac{\partial}{\partial g} \left(\sum_{a=1}^{\bar{M}^{(j)}} \frac{(\lambda_a^{(j)})^2}{2} + \sum_{a=M^{(j)}+1}^{N^{(j)}} \frac{(\lambda_a^{(j)})^2}{2} \right. \\
&\qquad \left. + \left\langle \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right| H \left| \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right\rangle \right) \\
&\simeq \frac{\partial}{\partial g} \left\langle \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right| H \left| \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right\rangle.
\end{aligned} \tag{B18}
$$

Summing over all the cells, this gives

$$
\begin{aligned}
&C_{{\rm c},>\Lambda} := \\
&2g^2 \frac{\partial}{\partial g} \left(\frac{1}{L} \sum_{j=1}^{m} \left\langle \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right| H \left| \boldsymbol{\lambda}^{(j)}_{\bar{M}^{(j)}+1, N^{(j)}-M^{(j)}} \right\rangle \right) \\
&\simeq 2g^2 \frac{\partial}{\partial g} \left(\frac{1}{L} \sum_{j=1}^{m} \langle \boldsymbol{\lambda}_{N^{(j)}} |\, H \,| \boldsymbol{\lambda}_{N^{(j)}} \rangle \right) =: C_{\rm c}.
\end{aligned} \tag{B19}
$$

Plugging this into Eq. (B17) we get the final result

$$
w(p) \underset{|p|\to\infty}{\simeq} \frac{C_{\rm c} + C_{\rm r}}{p^4}, \tag{B20}
$$

which is our Eq. (4) in the main text.

Appendix C: Calculation of the product $g^2 g^{(2)}(0)$ in the $g \to \infty$ limit

In the main text, we use the relation

$$
\lim_{g\to\infty} n^2 g^2 g^{(2)}(0) = 8\hbar^2/m\,[ne - j^2/(2m)], \tag{C1}
$$

where $n = \int \rho(q)dq$ is the particle density, $j = \int q\rho(q)dq$ is the momentum density, and $e = \int q^2/(2m)\rho(q)dq$ is the energy density in a state of arbitrary rapidity density $\rho(q)$. This identity can be derived as follows. We first consider finite g. The Hellmann-Feynman theorem, together with thermodynamic Bethe Ansatz calculations (see e.g. Ref. [?], or the supplementary methods of Ref. [?]), lead to the following formula for $g^{(2)}(0)$, or equivalently for the density of interaction energy $e_{\rm I} := g\partial(E/L)/\partial g$:

$$
e_{\rm I} \;=\; \frac{1}{2} n^2\, g\, g^{(2)}(0) \;=\; \int \left[q/m - v^{\rm eff}(q) \right] q\, \rho(q) dq. \tag{C2}
$$

Here $v^{\rm eff}(q)$ is the 'effective velocity' defined by the thermodynamic Bethe Ansatz formula

$$
v^{\rm eff}(q) \;=\; \frac{1}{m} \frac{{\rm id}^{\rm dr}(q)}{1^{\rm dr}(q)}, \tag{C3}
$$

where ${\rm id}(q) = q$, $1(q) = 1$, and the 'dressing' of a function $f(q)$ is defined as

$$
f^{\rm dr}(q) \;=\; f(q) + \int \varphi(q-q') \frac{f^{\rm dr}(q')}{1^{\rm dr}(q')} \rho(q') dq'. \tag{C4}
$$

Here $\varphi(q) = 2mg/((mg/\hbar)^2 + q^2)$ is the Lieb-Liniger kernel [? ?]. Expanding at first order in $1/g$, one finds $1^{\rm dr}(q) = 1 + 2\hbar^2 n/(mg) + O(1/g^2)$ and ${\rm id}^{\rm dr}(q) = q + 2\hbar^2 j/(mg) + O(1/g^2)$, so

$$
v^{\rm eff}(q) \underset{g\to\infty}{=} \frac{q}{m} - \frac{2\hbar^2}{m^2 g}(qn - j) + O(1/g^2). \tag{C5}
$$

Inserting this into Eq. (C2), one gets the relation (C1).

Appendix D: Evolution of the atom density, momentum density and energy density under one-body losses in the hard-core limit

In the main text we use the fact that, in the hard-core limit, the atom density, momentum density and energy density evolve with time as $n(t) = e^{-Gt} n_0$, $j(t) = e^{-Gt} j_0$, $e(t) = e^{-Gt} e_0$ respectively.

This can be derived using the results of Ref. [?] (see also the related Ref. [?] for the much more difficult case of finite g). First, one uses the rapidity distribution to define a generating function for the conserved charges (following Ref. [?]),

$$Q(z) := \frac{i}{\pi} \int \frac{\rho(q) dq}{z - q}, \tag{D1}$$

for $z \in \mathbb{C}$, $\mathrm{Im}\, z > 0$. $Q(z)$ is analytic for $\mathrm{Im}\, z > 0$. Moreover, for q real, we have

$$\lim_{z \to q} \mathrm{Re}[Q(z)] \; = \; \rho(q). \tag{D2}$$

Under losses, $Q(z)$ evolves in time. At time t, and in terms of the initial rapidity distribution $\rho_0(\lambda)$, it is equal to [?]

$$Q(z) \; = \; \frac{\frac{i\, e^{-Gt}}{\pi \hbar} \int \frac{\rho_0(\lambda) d\lambda}{(z-\lambda)/\hbar + 2 i n_0 (1 - e^{-Gt})}}{1 - i 2 (1 - e^{-Gt}) \int \frac{\rho_0(\lambda) d\lambda}{(z-\lambda)/\hbar + 2 i n_0 (1 - e^{-Gt})}}, \tag{D3}$$

for $\mathrm{Im}\, z > 0$.

The atom density $n = \int \rho(q) dq$, the momentum density $j = \int q \rho(q) dq$ and the energy density $e = \int q^2 \rho(q) dq / (2m)$ appear in the asymptotic expansion of Eq. (D1) at large z:

$$Q(z) \underset{z \to \infty}{=} \frac{i}{\pi} \left(\frac{n}{z} + \frac{j}{z^2} + \frac{2me}{z^3} + \dots \right) \tag{D4}$$

Expanding Eq. (D3) to order $O(1/z^3)$, one finds

$$Q(z) \underset{z \to \infty}{=} \frac{i}{\pi} \left(\frac{e^{-Gt} n_0}{z} + \frac{e^{-Gt} j_0}{z^2} + \frac{2m\, e^{-Gt} e_0}{z^3} + \dots \right), \tag{D5}$$

which gives the time-dependence claimed above for the three densities.

Appendix E: Bogoliubov theory in the quasicondensate regime (after Mora and Castin)

We follow the conventions of Mora and Castin [?]. Inserting a phase-amplitude representation of the annihilation operator, $\Psi(z) = \sqrt{n + \delta n} e^{i\theta}$ with $[\delta n(z), \theta(z')] = i \delta(z - z')$, in the Hamiltonian (2), one finds to second order:

$$H - \mu N \simeq \int \left[\frac{\hbar^2}{8mn} (\partial_z \delta n)^2 + \frac{g}{2} \delta n^2 + \frac{\hbar^2 n}{2m} (\partial_z \theta)^2 \right] dz.$$

This quadratic Hamiltonian allows to grasp quantum fluctuations around the classical profile which solves the Gross-Pitaevski equation, $n = N/L = \mu/g$ where μ is the chemical potential. One can define a boson annihilation field $B(z) = \frac{1}{2\sqrt{n}} \delta n(z) + i \sqrt{n} \theta(z)$ such that $[B(z), B^\dagger(z')] = \delta(z - z')$, and its Fourier modes $B_q = \int e^{-iqz/\hbar} B(z) dz / \sqrt{L}$ with $q \in (2\pi\hbar/L)\mathbb{Z}$. Then the quadratic Hamiltonian becomes, up to constant terms,

$$H - \mu N \; \simeq \; \frac{1}{2} \sum_q \begin{pmatrix} B_q \\ B^\dagger_{-q} \end{pmatrix}^\dagger \begin{pmatrix} \frac{q^2}{2m} + \mu & \mu \\ \mu & \frac{q^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} B_q \\ B^\dagger_{-q} \end{pmatrix},$$

where we have used $\mu = gn$. Finally, the Hamiltonian H_q is diagonalized by a Bogoliubov transformation

$$\begin{pmatrix} B_q \\ B^\dagger_{-q} \end{pmatrix} = \begin{pmatrix} \bar{u}_q & \bar{v}^*_q \\ \bar{v}_{-q} & \bar{u}^*_{-q} \end{pmatrix} \begin{pmatrix} b_q \\ b^\dagger_{-q} \end{pmatrix}$$

with $|\bar{u}_q|^2 - |\bar{v}_q|^2 = 1$. Here a convenient choice is $\bar{u}_q = \bar{u}^*_q = \cosh(\theta_q/2)$ and $\bar{v}_q = \bar{v}^*_q = -\sinh(\theta_q/2)$ with $\tanh \theta_q = \mu / (\mu + \frac{q^2}{2m})$, which gives

$$H - \mu N \; \simeq \; \sum_q \varepsilon_q b^\dagger_q b_q \; + \; \mathrm{const.},$$

with a dispersion relation $\varepsilon_q = \sqrt{\frac{q^2}{2m} \left(\frac{q^2}{2m} + \mu \right)}$.

1. Population of Bogoliubov modes and momentum distribution

Let us consider a state where the population of each Bogoliubov mode is $\alpha_q = \langle b^\dagger_q b_q \rangle$. The one-particle density matrix is (see Ref. [?], formula (184)):

$$g^{(1)}(z) \; = \; \exp \left[-\frac{1}{n} \int \frac{dq}{2\pi\hbar} [(\bar{u}_q^2 + \bar{v}_q^2) \alpha_q + \bar{v}_q^2] (1 - \cos(qz/\hbar)) \right]. \tag{E1}$$

Following Lieb [?], we identify quasiparticle excitations with large rapidities with the large-q Bogoliubov modes. Then we are interested in the case when n_q decays as $2\pi\hbar C_{\rm r}/q^4$ at large q, where $C_{\rm r}$ is the same constant as in the main text. We note that

$$[(\bar{u}_q^2 + \bar{v}_q^2) \alpha_q + \bar{v}_q^2] \underset{q \to \infty}{\simeq} 2\pi\hbar \frac{C_{\rm r} + C_{\rm c}}{q^4}, \tag{E2}$$

which follows from the fact that $\bar{v}_q^2 = \bar{u}_q^2 - 1 = m^2 \mu^2 / q^4 + O(1/q^6)$, and $m^2 \mu^2 = 2\pi\hbar C_{\rm c}$ (valid in the quasicondensate regime). In general, $1/k^4$ tails result in a discontinuity of the third derivative of the Fourier transform, according to $\partial_x^3 \left(\int \frac{dk}{2\pi} \frac{e^{ikx}}{k^4 + \epsilon^4} \right)_{|_{x \to 0^+}} - \partial_x^3 \left(\int \frac{dk}{2\pi} \frac{e^{ikx}}{k^4 + \epsilon^4} \right)_{|_{x \to 0^-}} =$

1. Thus, the discontinuity of the argument of the exponential in (E1) is

$$\partial_z^3 \left(\frac{1}{n} \int \frac{dq}{2\pi\hbar} [(\bar{u}_q^2 + \bar{v}_q^2)\alpha_q + \bar{v}_q^2](1 - \cos(qz/\hbar)) \right)_{|_{z\to 0^+}}$$
$$-\partial_z^3 \left(\frac{1}{n} \int \frac{dq}{2\pi\hbar} [(\bar{u}_q^2 + \bar{v}_q^2)\alpha_q + \bar{v}_q^2](1 - \cos(qz/\hbar)) \right)_{|_{z\to 0^-}}$$
$$= -2\pi \frac{C_\mathrm{r} + C_\mathrm{c}}{\hbar^3 n}.$$

Consequently, $g^{(1)}(z)$ also possesses a discontinuity in its third derivative,

$$\partial_z^3 g^{(1)}_{|_{z\to 0^+}} - \partial_z^3 g^{(1)}_{|_{z\to 0^-}} = 2\pi \frac{C_\mathrm{r} + C_\mathrm{c}}{\hbar^3 \rho_0}. \tag{E3}$$

Taking the Fourier transform, one finds that the momentum distribution has a tail with coefficient $C_\mathrm{r} + C_\mathrm{c}$, as claimed in the main text:

$$\begin{aligned} w(p) &= \frac{n}{2\pi\hbar} \int_0^L e^{ipz/\hbar} g^{(1)}(z) dz \\ &\underset{p\to\infty}{\simeq} (C_\mathrm{r} + C_\mathrm{c})/p^4. \end{aligned} \tag{E4}$$

2. The effect of losses on Bogoliubov modes

The effect of losses in the quasicondensate regime has been investigated in Refs. [? ? ? ?]. For the convenience of the reader, we recall the results that are useful for this Letter.

In terms of the Fourier modes of the phase and density fluctuation fields, $\theta_q = (1/\sqrt{L}) \int dz \theta(z) e^{-iqz/\hbar}$ and $\delta n_q = (1/\sqrt{L}) \int dz \delta n(z) e^{-iqz/\hbar}$, the population α_q of the Bogoliubov mode q reads

$$\alpha_q = \frac{f_q}{4n} \langle \delta n_{-q} \delta n_q \rangle + \frac{n}{f_q} \langle \theta_{-q} \theta_q \rangle - \frac{1}{2}, \tag{E5}$$

where $f_q = \sqrt{(q^2/(2m) + 2gn)/(q^2/(2m))}$.

Under losses, the density n and the coefficient f_q become time-dependent, as well as the phase and density fluctuations $\langle \delta n_{-q} \delta n_q \rangle$ and $\langle \theta_{-q} \theta_q \rangle$. One finds

$$\begin{aligned} \frac{d\alpha_q}{dt} &= \frac{f_q}{4n} \frac{d\langle \delta n_{-q} \delta n_q \rangle}{dt} + \frac{n}{f_q} \frac{d\langle \theta_{-q} \theta_q \rangle}{dt} \\ &+ \frac{1}{f_q/n} \frac{d(f_q/n)}{dt} \left[\frac{f_q}{4n} \langle \delta n_{-q} \delta n_q \rangle - \frac{n}{f_q} \langle \theta_{-q} \theta_q \rangle \right]. \end{aligned} \tag{E6}$$

We are assuming slow losses. Then, to compute $d\alpha_q/dt$, which is a slowly varying quantity, one can average over a time $2\pi/\varepsilon_q$. This time-average ensures equipartition of energy between the two conjuagte variables δn_q and θ_{-q}. Consequently, the second line in the equation vanishes, and we have

$$\frac{d\alpha_q}{dt} = \frac{f_q}{4n} \frac{d\langle \delta n_{-q} \delta n_q \rangle}{dt} + \frac{n}{f_q} \frac{d\langle \theta_{-q} \theta_q \rangle}{dt}, \tag{E7}$$

which is the equation used in the main text. Note that the fact that $d\alpha_q/dt$ is not affected by the slow time evolution of n and f_q (*i.e.* the vanishing of the second line of Eq. (E6)) can also be interpreted as the result of adiabatic following of the eigenstates of $H_q = \varepsilon_q(b_q^+ b_q + 1/2)$. We now recall the effect of losses on density and phase fluctuations, analyzed in Refs. [?].

a. *Effet of losses on density fluctuations*

The goal of this section is to derive the formula for the evolution of the density fluctuations,

$$\begin{aligned} \frac{d\langle \delta n(z) \delta n(z') \rangle}{dt} &= K^2 G n^K \delta(z - z') \\ &\quad -2K^2 G n^{K-1} \langle \delta n(z) \delta n(z') \rangle, \end{aligned} \tag{E8}$$

which is used in the main text.

To do this, we consider a cell of length ℓ, much smaller than the typical length scale of variation of the phase θ, but large enough so that it contains a number of atoms $N \gg 1$. We note $\bar{N} = n\ell$ the atom number corresponding to the mean atomic density n in the gas. We are interested in the effect of losses during a time interval Δt satisfying $\bar{N}^{-K} \ll \gamma \Delta t \ll \bar{N}^{1-K}$ where $\gamma := G/\ell^{K-1}$ is the loss rate in the cell. This ensures that the number of lost atoms is much larger than one, but much smaller than $\bar{N}$.

We consider an initial state with an atom number distribution $P_0(N)$. Here fluctuations can be either of statistical or of quantum nature. Let $\mathcal{P}_0(M)$ the probability to have M loss events until time Δt. One has

$$\mathcal{P}_0(M) = \sum_N P_0(N) P(M|N), \tag{E9}$$

where $P(M|N)$ is the probability to have M loss events conditioned to an initial number of atoms N. Under the assumption $\gamma \Delta t \ll \bar{N}^{1-K}$, this is well approximated by a Poisson distribution [?]

$$P(M|N) = \frac{1}{M!} e^{-\gamma \Delta t \, N^K} \left(\gamma \Delta t \, N^K \right)^M. \tag{E10}$$

Furthermore, for $\gamma \Delta t \gg \bar{N}^{-K}$, the Poissonian becomes a Gaussian,

$$P(M|N) \simeq \frac{e^{-(M - N^K \gamma \Delta t)^2}}{\sqrt{2\pi}\sigma}. \tag{E11}$$

The variance can be approximated by its value for $N = \bar{N}$, which is

$$\sigma = \sqrt{\gamma \Delta t \, \bar{N}^K}. \tag{E12}$$

The probability to have N atoms in the cell at time Δt is then

$$\begin{aligned} P(N) &= \sum_M P_0(N + KM) P(M|N + KM) \\ &\simeq \int dM P_0(N + KM) P(M|N + KM), \end{aligned} \tag{E13}$$

where we have used the fact that both N and M are typically large to replace the sum by an integral.

We are now ready to compute the atom number fluctuations at time Δt. For this we introduce $\delta N(0) = N - \bar{N}$ at time 0, and $\delta N(\Delta t) = N(\Delta t) - \bar{N}(\Delta t)$ at time Δt, where $\bar{N}(\Delta t) = \bar{N} - K\gamma\Delta t\, \bar{N}^K$ is the atom number corresponding to the gas mean density after Δt. Using (E13), one gets

$$\langle \delta N(\Delta t)^2 \rangle = \int dN \int \begin{array}{l} dM(N - \bar{N}(\Delta t))^2 \\ P_0(N+KM)P(M|N+KM) \end{array}$$

With the change of variable $\tilde{N} = N + KM$, this becomes

$$\langle \delta N(\Delta t)^2 \rangle = \int d\tilde{N} P_0(\tilde{N}) \int dM(\tilde{N} - KM - \bar{N}(\Delta t))^2 P(M|\tilde{N}). \tag{E14}$$

Then the Gaussian approximation of $P(M|\tilde{N})$ (Eq. (E11)) gives

$$\langle \delta N(\Delta t)^2 \rangle = K^2\gamma\Delta t\, \bar{N}^K + \int d\tilde{N} P_0(\tilde{N})(\tilde{N} - K\gamma\Delta t\, \tilde{N}^K - \bar{N} + K\gamma\Delta t\, \bar{N}^K)^2. \tag{E15}$$

Using the fact that the atom number fluctuations around $\bar{N}$ are small, one can expand to lowest order in $\delta N(0) = \tilde{N} - \bar{N}$. Then the expression inside the parenthesis becomes $(1 - K^2\gamma\Delta t\bar{N}^{K-1})\, \delta N(0)$; the square of that expression is $(1 - 2K^2\gamma\Delta t\bar{N}^{K-1})\, \delta N(0)^2$ at first order in $\gamma\Delta t\, \bar{N}^{K-1}$. Thus we obtain

$$\langle \delta N(\Delta t)^2 \rangle = K^2\gamma\Delta t\, \bar{N}^K + \left(1 - 2K^2\gamma\Delta t\, \bar{N}^{K-1}\right) \langle \delta N^2 \rangle. \tag{E16}$$

This lead to the differential form

$$\frac{d\langle \delta N^2 \rangle}{dt} = K^2 G n^K \ell - 2K^2 G n^{K-1} \langle \delta N^2 \rangle, \tag{E17}$$

where we have used $\gamma \bar{N}^{K-1} = Gn^{K-1}$.

Let us now consider two differents cells located around z_α and z_β. For given atom numbers N_α and N_β in the cell located in z_α and z_β respectively, the fluctuations of the number of loss events in both cells are not correlated. Then similar calculations as above give

$$\frac{d\langle \delta N_\alpha \delta N_\beta \rangle}{dt} = -2K^2\gamma\Delta t\, \bar{N}^{K-1} \langle \delta N_\alpha \delta N_\beta \rangle. \tag{E18}$$

Eq.(E17) and (E18) imply that the evolution of the fluctuations of the density field $\delta n(z) \simeq \delta N/\ell$ (for a cell around at position z) is given by Eq. (E8) as claimed.

b. *Effect of losses on phase fluctuations*

Although losses do not depend on the phase variable, losses do have an impact on the phase fluctuations $\langle \theta(z)^2 \rangle$. This is due to the broadening of the phase as one gains knowledge on the atom number N, its conjugate variable. This ensures the preservation of quantum uncertainty relations. Losses increase our knowledge of N because if one records the losses, then one gains knowledge on N [This effect can be exploited in a feedback scheme to cool down the Bogoliubov modes [**?**]]. The quantitative evaluation of this effect is done in Ref. [**?**], and the result reads:

$$\frac{d\langle \theta(z)\theta(z') \rangle}{dt} = \frac{1}{4} K^2 G n^{K-2} \delta(z - z'). \tag{E19}$$

This is the equation used in the main text. [We point out that Eq. (E19), as well as Eq. (E8), can also be derived from stochastic equations, see Ref. [**?**].]

c. *Evolution of the population of the Bogoliubov modes*

Taking the Fourier transform of Eq. (E8) and Eq. (E19) and injecting into Eq. (E7) we find

$$d\alpha_q/dt = K^2 G n^{K-1} \left(-\alpha_q - 1/2 + 1/4(f_q + f_q^{-1})\right). \tag{E20}$$

This equation, together with the equation $n = n_0 e^{-Gt}$, allows to compute $\alpha_q(t)$. This equation is valid for any value of q.

3. Evolution of the momentum distribution

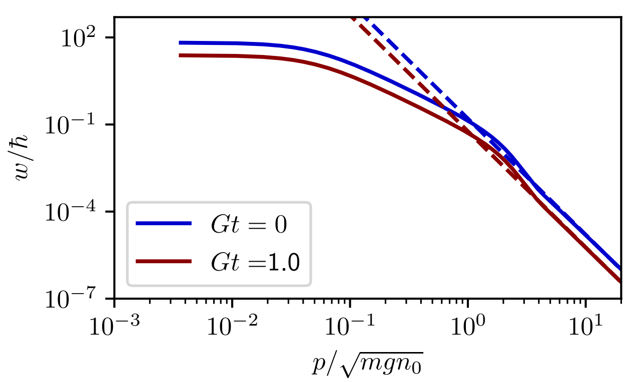

FIG. 2. Momentum distribution of a quasicondensate submitted to one-body losses of rate G. The initial state is a thermal state at a linear density $n_0 = 10\sqrt{mgn_0}/\hbar$ and at a temperature $T = gn_0$. Its momentum distribution is shown as the blue solid line. The dashed blue line is $C_{c,0}/p^4$, where $C_{c,0} = (mn_0 g)^2/(2\pi)$ is the initial contact density ($g^{(2)}(0) \simeq 1$ in the quasicondensate regime). The red solid line is the momentum distribution after a time $t = 1/G$. The dashed red line is $C(t)/p^4$, where $C(t) = e^{Gt}(mn_0 g)^2/(2\pi)$.

We performed numerical calculations for one-body losses ($K = 1$), starting from a thermal state with linear density n_0 and temperature T. We use Eq.eq:dalphadtvsfq, injecting $n(t) = n_0 e{-}Gt$, to compute $\alpha_q(t)$ for all q. We then compute the first order correlation function using Eq.(E1). We finally take its Fourier transform to extract the momentum distribution $w(p)$. Fig.2 shows resulting momentum distributions, in log-log scale, at time $t = 0$ and at time $t = 1/\Gamma$. We see that, for those parameters, the $1/p^4$ behavior appears for momenta larger than $\simeq 3\sqrt{mgn_0}$. The amplitude of the tails is in agreement with the analytic prediction $C(t) = e^{Gt}(mn(t)g)^2/(2\pi)$.

4. Solution of the differential equation (13) for losses in the quasicondensate regime

We use the dimensionless variable $\tau = Kn_0^{K-1}Gt$, where n_0 is the atom density at $t = 0$. In the quasi-condensate regime, we have $g^{(K)}(0) = 1$, so the atom density $n(\tau)$ evolves according to

$$\frac{d(n/n_0)}{d\tau} = -(n/n_0)^K. \tag{E21}$$

The differential equation (13) in the main text is

$$\frac{dC_{\rm r}}{d\tau} = -K(n/n_0)^{K-1}C_{\rm r} + KC_{\rm c,0}\,(n/n_0)^{K+1}, \tag{E22}$$

with $C_{\rm c,0} = m^2g^2n_0^2/(2\pi\hbar)$. Using Eq. (E21) one can easily check that the solutions of that differential equation are (for $K \neq 2$)

$$C_{\rm r}(\tau) = \frac{K\,C_{\rm c,0}}{K-2}(n/n_0)^2 + A\,(n/n_0)^K, \tag{E23}$$

for any constant A. The constant A is then fixed in terms of the initial condition $C_{\rm r}(t = 0) = 0$ (this is the initial condition assumed in the main text). This gives (for $K \neq 2$):

$$C_{\rm r}(\tau) = \frac{K\,C_{\rm c,0}}{K-2}(n/n_0)^2\left[1-(n/n_0)^{K-2}\right]. \tag{E24}$$

If $K = 2$, then we have instead

$$(K=2) \quad C_{\rm r}(\tau) = -2C_{\rm c,0}(n/n_0)^2\log(n/n_0). \tag{E25}$$

Recall that $C_{\rm c}(\tau) = m^2g^2n(\tau)^2/(2\pi\hbar)$. Then we get

$$\frac{C_{\rm r}(\tau)}{C_{\rm c}(\tau)} = \begin{cases} K/(K-2)\left[1-(n/n_0)^{K-2}\right] & \text{if} \quad K \neq 2, \\ -2\log(n/n_0) & \text{if} \quad K = 2. \end{cases} \tag{E26}$$

Finally, we note that the solution of Eq. (E21) is

$$\frac{n(\tau)}{n_0} = \begin{cases} \left[1+(K-1)\tau\right]^{1/(1-K)} & \text{if} \quad K > 1, \\ e^{-\tau} & \text{if} \quad K = 1. \end{cases} \tag{E27}$$

Eqs. (E26) and (E27) give the large τ behavior reported in Eq. (14) in the main text.

Appendix F: Generalization to non-uniform gases

In most experimental situations, gases are confined into a slowly-varying longitudinal potential, often of quadratic form. The confinement is however usually weak enough to ensure the validity of the Generalized Hydrodynamics approach [? ?] (which corresponds, in the case of stationary states, to the well known Local Density Approximation). The rapidity distribution then becomes a two dimensional function $\rho(q, z)$, where, for a given z, $\rho(q, z)$ is the local rapidity distribution. The coefficient $C_r = \lim_{q\to\infty} q^4\rho(q)$ becomes z-dependent and we note it $C_r(z)$. Moreover we introduce the extensive quantity $W(p) = \int dz\ w(p, z)$, where $w(p, z)$ is the local momentum distribution, and $\mathcal{C} = \lim_{p\to\infty} p^4W(p)$. $W(p)$ is normalized to $\int dp\ W(p) = N$ where N is the total atom number. Eq. (4) of the main text then becomes

$$\mathcal{C} = \int dz\,(C_{\rm c}(z) + C_r(z)) \tag{F1}$$

where $C_{\rm c}(z) = m^2g^2n(z)^2g^{(2)}(0, z)/(2\pi\hbar)$ is the local contact density. Here $g^{(2)}(0, z) = \langle\psi^+(z)\psi^+(z)\psi(z)\psi(z)\rangle$ is the zero-distance two-body correlation function, computed at position z. For a given z, $C_{\rm c}(z)$ is a functional of $\rho(p, z)$, see Eq. (C2). Thus C can be computed once the function $\rho(p, z)$ is known.

As losses occur, $\rho(p, z)$ is locally modified by losses. The system is then, in general, brought to a non-stationary solution of the Generalized Hydrodynamics equations and one should compute the time-evolution of $\rho(p, z)$ using Eq. (16) of Ref. [?].